APPARENT MODULUS OF A COMPOSITE BEAM

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INTRODUCTION:

Let us consider a composite beam of thickness t, consisting of three different layers of moduli E_1, E_2 and E_3 respectively, the thicknesses of each layer being t_1, t_2 and t_3 as shown in figure 1.

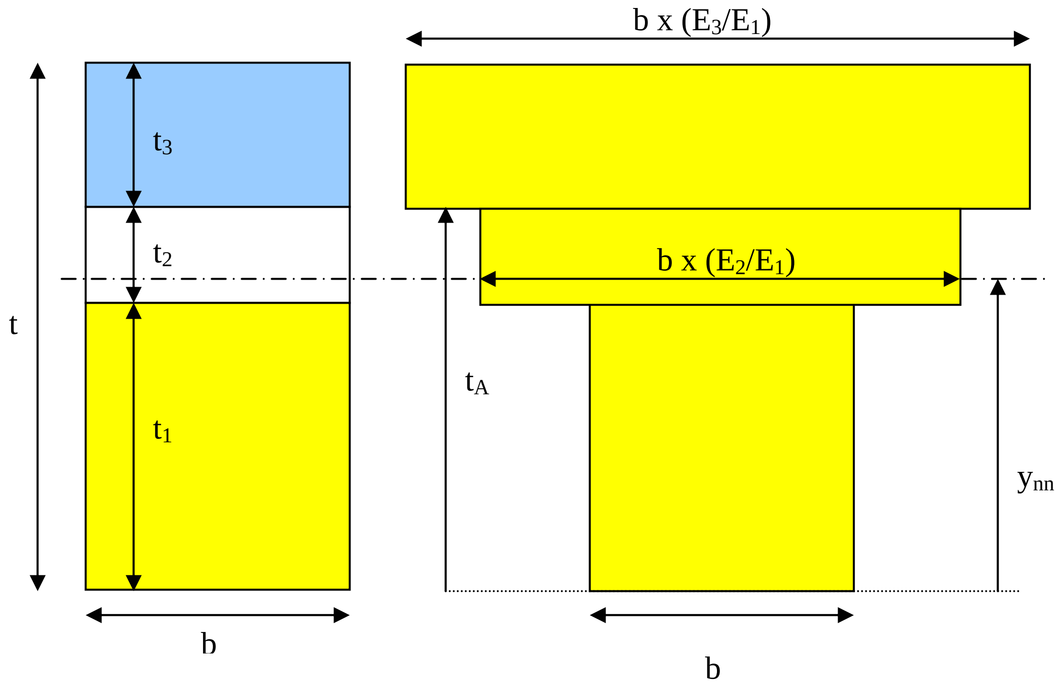

Figure1. Schematic of a composite beam showing equivalent transformed section.

POSITION OF THE NEUTRAL AXIS:

When bending a cross section, which is not symmetrical about the centroid, the neutral axis will no longer be in the central line. However, there must exist a line along the cross section, where the length does not change. The later can be found found from the requirement that the following integral must be zero [1]:

$$\int_0^t (y - y_{nn})dA = 0 \qquad (1)$$

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Substituting the width in the respective parts of the cross section yields:

$$\int_0^{t_1}(y-y_{nn})bdy+\int_{t_1}^{t_A}(y-y_{nn})\frac{E_2}{E_1}bdy+\int_{t_A}^{t}(y-y_{nn})\frac{E_3}{E_1}bdy=0$$

$$\int_0^{t_1}ydy-y_{nn}\int_0^{t_1}dy+\frac{E_2}{E_1}\int_{t_1}^{t_A}ydy-\frac{E_2}{E_1}y_{nn}\int_{t_1}^{t_A}dy+\frac{E_3}{E_1}\int_{t_A}^{t}ydy-\frac{E_3}{E_1}y_{nn}\int_{t_A}^{t}dy=0$$

$$\frac{t_1^2}{2}-y_{nn}t_1+\frac{E_2}{E_1}\left[\frac{(t_A^2-t_1^2)}{2}-y_{nn}(t_A-t_1)\right]+\frac{E_3}{E_1}\left[\frac{(t^2-t_A^2)}{2}-y_{nn}(t-t_A)\right]=0$$

$$y_{nn}\left[\frac{E_1t_1+E_2(t_A-t_1)+E_3(t-t_A)}{E_1}\right]=\frac{1}{2}\left[\frac{E_1t_1^2+E_2(t_A^2-t_1^2)+E_3(t^2-t_A^2)}{E_1}\right]$$

$$y_{nn}=\frac{1}{2}\left[\frac{E_1t_1^2+E_2(t_A^2-t_1^2)+E_3(t^2-t_A^2)}{E_1t_1+E_2(t_A-t_1)+E_3(t-t_A)}\right]$$

$$\because t=t_1+t_2+t_3, t_A=t_1+t_2$$

$$y_{nn}=\frac{1}{2}\left[\frac{E_1t_1^2+E_2\left\{(t_1+t_2)^2-t_1^2\right\}+E_3\left\{(t_1+t_2+t_3)^2-(t_1+t_2)^2\right\}}{E_1t_1+E_2t_2+E_3t_3}\right]$$

Extending this for an ‘n’ layered laminate, we get:

$$y_{nn}=\frac{1}{2}\left[\frac{E_1t_1^2+E_2\left\{(t_1+t_2)^2-t_1^2\right\}+\cdots+E_r\left\{(t_1+t_2+\cdots+t_r)^2-(t_1+t_2+\cdots+t_{r-1})^2\right\}+\cdots+E_n\left\{(t_1+t_2\cdots+t_n)^2-(t_1+t_2+\cdots+t_{n-1})^2\right\}}{E_1t_1+E_2t_2+\cdots+E_rt_r+\cdots+E_nt_n}\right]$$

$$y_{nn}=\frac{1}{2}\left[\frac{E_1t_1^2+E_2t_2\left\{2t_1+t_2\right\}+E_3t_3\left\{2(t_1+t_2)+t_3\right\}+\cdots+E_rt_r\left\{2(t_1+t_2+\cdots+t_{r-1})+t_r\right\}+\cdots+E_nt_n\left\{2(t_1+t_2+\cdots+t_{n-1})+t_n\right\}}{E_1t_1+E_2t_2+\cdots+E_rt_r+\cdots+E_nt_n}\right]$$

This can also be written as:

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$$y_{nn} = \frac{1}{2}\frac{\sum_{r=1}^{n} E_r t_r \left\{2\left(\sum_{i=1}^{r} t_{i-1}\right) + t_r\right\}}{\sum_{j=1}^{n} E_j t_j}, put: t_0 = 0 \qquad (2)$$

THE SECOND MOMENT OF AREA:

The second moment of area is given by the integral [1]:

$$I = \int_0^t (y - y_{nn})^2 dA \qquad (3)$$

while equation 2 is related to the absence of a net normal force, this equation stems from the fact that the overall action of the bending stresses in the cross section must be equal to the applied bending moment, *M*. Thus the above integral can be expressed as a sum of two integrals over the regions of different width –

$$\frac{I}{b} = \int_0^{t_1} (y - y_{nn})^2 dy + \int_{t_1}^{y_{nn}} (y - y_{nn})^2 \frac{E_2}{E_1} dy + \int_{y_{nn}}^{t_A} (y - y_{nn})^2 \frac{E_2}{E_1} dy + \int_{t_A}^{t} (y - y_{nn})^2 \frac{E_3}{E_1} dy$$

$$\frac{3I}{b} = (t_1 - y_{nn})^3 + y_{nn}^3 + \frac{E_2}{E_1}\left[(t_A - y_{nn})^3 - (t_1 - y_{nn})^3\right] + \frac{E_3}{E_2}\left[(t - y_{nn})^3 - (t_A - y_{nn})^3\right]$$

$$\frac{3I}{b} = y_{nn}^3 + \left(1 - \frac{E_2}{E_1}\right)(t_1 - y_{nn})^3 + \left(\frac{E_2}{E_1} - \frac{E_3}{E_1}\right)(t_A - y_{nn})^3 + \frac{E_3}{E_1}(t - y_{nn})^3$$

The final expression for the second moment of area is given by:

$$I = \frac{b}{3}\left[y_{nn}^3 + \left(1 - \frac{E_2}{E_1}\right)(t_1 - y_{nn})^3 + \left(\frac{E_2}{E_1} - \frac{E_3}{E_1}\right)\left\{(t_1 + t_2) - y_{nn}\right\}^3 + \frac{E_3}{E_1}(t - y_{nn})^3\right]$$

Extending this for an ‘n’ layered laminate, we get:

$$I = \frac{b}{3}\left[y_{nn}^3 + \left(1 - \frac{E_2}{E_1}\right)(t_1 - y_{nn})^3 + \cdots\cdots + \left(\frac{E_{r-1}}{E_1} - \frac{E_r}{E_1}\right)\left\{(t_1 + t_2 + \cdots + t_{r-1}) - y_{nn}\right\}^3 + \cdots + the, nth\_term\right]$$

This can be written as:

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$$I = \frac{b}{3}\left[y_{nn}^{3} + \frac{1}{E_1}\sum_{r=2}^{n}\left[(E_{r-1} - E_r)\left\{ \sum_{i=1}^{r-1} t_i - y_{nn} \right\}^{3} \right]\right] \tag{4}$$

MODULUS OF THE COMPOSITE:

The apparent modulus of the composite with ‘n’ layers can be obtained from the equation:

$$E_{app} I_{app} = E_1 I \tag{5}$$

$$I_{app} = \frac{bh^3}{12} \tag{6}$$

From equations 4 – 6, we get:

$$E_{app} = \frac{E \cdot \frac{b}{3}\left[y_{nn}^{3} + \frac{1}{E_1}\sum_{r=2}^{n}\left[(E_{r-1} - E_r)\left\{ \sum_{i=1}^{r-1} t_i - y_{nn} \right\}^{3} \right]\right]}{\frac{bh^3}{12}}$$

$$E_{app} = \frac{4b\left[E_1 y_{nn}^{3} + \sum_{r=2}^{n}\left[(E_{r-1} - E_r)\left\{ \sum_{i=1}^{r-1} t_i - y_{nn} \right\}^{3} \right]\right]}{h^3}$$

REFERENCE:

1. Van Gemert, D. A. and G. De Roeck, *Beginselen van sterkteleer*. 1988, Leuven: L. Wouters.

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