Elastic Bending of Beams

The basic differential equation used for most of the bean bending problems is -

$$EI\frac{d^2y}{dx^2} = M$$

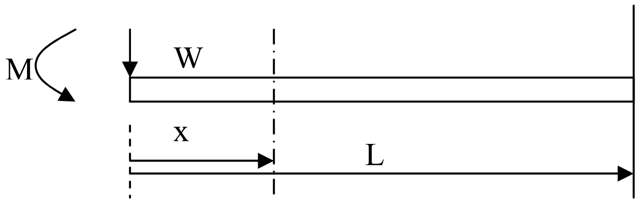
and

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A$$

$$y = \iint \left[\frac{M}{EI}dx\right]dx + Ax + B$$

Where, A and B are constants of integration evaluated from known conditions of slope and deflection for particular values of x.

Case I: Cantilever with concentrated load at the end



$$M = EI\frac{d^2y}{dx^2} = -Wx$$

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

$$EIy = -\frac{Wx^3}{6} + Ax + B$$

Now when

$$x = L, \frac{dy}{dx} = 0 \Rightarrow A = \frac{WL^2}{2}$$

$$x = L, y = 0 \Rightarrow B = \frac{WL^3}{6} - \frac{WL^2}{2}L = -\frac{WL^3}{3}$$

$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6} + \frac{WL^2x}{2} - \frac{WL^3}{3}\right]$$

This gives the deflection at all values of x and produces a maximum value at the tip of the cantilever when x = 0,

$$y_{\max} = -\frac{WL^3}{3EI}$$

sp367@cam.ac.uk

The negative sign indicated that the deflection is in the negative y direction, i.e. downwards.

Similarly,

$$\frac{dy}{dx}=\frac{1}{EI}\left[-\frac{Wx^2}{2}+\frac{WL^2}{2}\right]$$

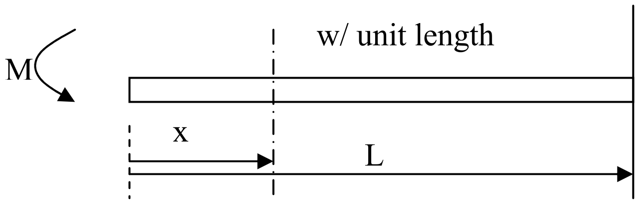
and produces a maximum value again when x = 0.

maximum slope is given by

$$\left(\frac{dy}{dx}\right)_{\max}=\frac{WL^2}{2EI}$$

which is positive.

Case II: Cantilever with uniformly distributed load



$$M=EI\frac{d^2y}{dx^2}=-\frac{wx^2}{2}$$

$$EI\frac{dy}{dx}=-\frac{wx^3}{6}+A$$

$$EIy=\frac{wx^4}{24}+Ax+B$$

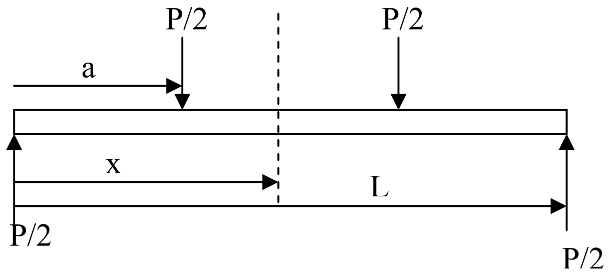
$$x=L,\frac{dy}{dx}=0,A=\frac{wL^3}{6}$$

$$x=L,y=0,B=\frac{wL^4}{24}-\frac{wL^4}{6}=-\frac{wL^4}{8}$$

$$y=\frac{1}{EI}\left[-\frac{wx^4}{24}+\frac{wL^3x}{6}-\frac{wL^4}{8}\right]$$

$$at\cdot x=0,y_{\max}=-\frac{wL^4}{8EI}and\left(\frac{dy}{dx}\right)_{\max}=\frac{wL^3}{6EI}$$

Case III: Four Point bending



From James M. Gere (Mechanics of Materials, 2001, page 895) one can find that the deflection at any point between the inner loading points is given by –

$$y = -\frac{Pa\left[3Lx - 3x^2 - a^2\right]}{12EI}$$

Putting, x = L/2 for maximum deflection and replacing y by δ (taking modulus only), we get

$$\delta = \frac{Pa\left[3L^2 - 4a^2\right]}{48EI}$$

Replacing E by the stress-strain relation, we get –

$$\delta = \frac{Pa\left[3L^2 - 4a^2\right]\varepsilon}{48I\,\dfrac{Mc}{I}}$$

Moment at a section along the centre of the beam is given by –

$$M = \frac{Pa}{2}$$

Taking c = h/2, we get –

$$\delta = \frac{Pa\left[3L^2 - 4a^2\right]\varepsilon}{48\dfrac{P}{2}\times\dfrac{h}{2}}$$

$$\delta = \frac{[3L^2 - 4a^2]\varepsilon}{12h}$$

$$\varepsilon = \frac{12\delta h}{\left[3L^2 - 4a^2\right]}$$

sp367@cam.ac.uk

The relation in the previous page is the general equation of strain at the surface (at the centre of the beam) of the flexural specimen under four point bending. Conventional four point bend test is done with two loading geometry, namely, a = L/3 and L/4. Thus, we have two geometrical situations.

Situation I. a = L/3

$$\varepsilon = \frac{12\delta h}{3L^2 - \frac{4L^2}{9}}$$

$$\varepsilon = \frac{108\delta h}{23L^2}$$

$$\varepsilon \sim \frac{4.7\delta h}{L^2}$$

Situation II. a = L/4

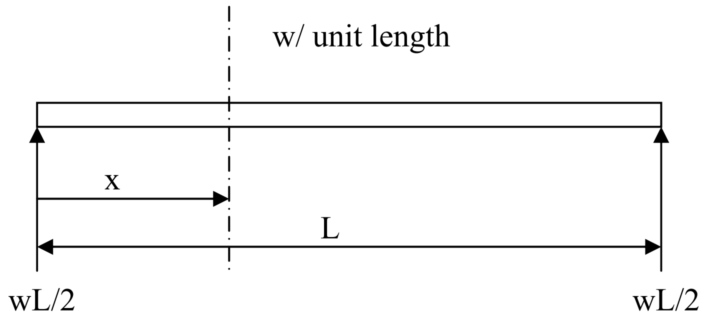
$$\varepsilon = \frac{12\delta h}{3L^2 - \frac{4L^2}{16}}$$

$$\varepsilon = \frac{192\delta h}{48L^2 - 4L^2}$$

$$\varepsilon = \frac{192\delta h}{44L^2}$$

$$\varepsilon \sim \frac{4.36\delta h}{L^2}$$

Case IV: Simply supported beam with uniformly distributed load



sp367@cam.ac.uk

$$
\begin{aligned}
&M = EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2} \\
&EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A \\
&EIy = \frac{wLx^3}{12} - \frac{wx^3}{24} + Ax + B \\
&at \cdot x = 0, y = 0 \Rightarrow B = 0 \\
&at \cdot x = L, y = 0 \Rightarrow \frac{wL^4}{12} + \frac{wL^4}{24} + AL \\
&\therefore A = -\frac{wL^4}{24} \\
&\Rightarrow y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]
\end{aligned}
$$

In this case the maximum deflection occurs at the centre of the beam where x = L/2.

$$
\therefore y_{\max} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]
$$

thus

$$
y_{\max} = -\frac{5wL^4}{384EI}
$$

For a beam of rectangular cross section $I = 1/12bh^3$

$$
y_{\max} = \frac{5wL^4}{32Ebh^3}
$$

Thus strain in the beam can be obtained form the above equation by using the relation between stress and strain.

$$
\sigma = E\varepsilon
$$

$$
E = \frac{\sigma}{\varepsilon}
$$

Putting the value of E in the previous relation, we get (taking the modulus of deflection only and putting $y_{max} = \delta$) –

$$\delta = \frac{5wL^4\varepsilon}{384\sigma I}$$

$$\delta = \frac{5wL^4\varepsilon}{384\frac{Mc}{I}I} \because \sigma = \frac{Mc}{I}$$

$$\delta = \frac{5wL^4\varepsilon}{384M\frac{h}{2}} \because c = \frac{h}{2} at.the.surface$$

Moment at x = L/2 is given by –

$$M = \frac{wL}{2} - \frac{wL^2}{8}$$

Therefore, the deflection equation transforms to –

$$\delta = \frac{5wL^3\varepsilon}{96\left[w - \frac{wL}{4}\right]h}$$

$$\delta = \frac{5L^3\varepsilon}{96\left[1 - \frac{L}{4}\right]h}$$

$$\varepsilon = \frac{96\left[1 - \frac{L}{4}\right]h\delta}{5L^3}$$

This expression gives the strain in the simply supported beam with uniformly distributed load.